Higher-Spin Geometry and String Theory

Dario Francia

Dipartimento di Fisica, Università di Roma Tre and INFN, Sezione di Roma III
Via della Vasca Navale 84, I-00146 Roma Italy
and
Max Planck Institute for Gravitational Physics (Albert Einstein Institute)
Am Mühlenberg 1 D-14476 Potsdam Germany †

E-mail: `francia@fis.uniroma3.it, Dario.Francia@aei.mpg.de`

Augusto Sagnotti

Ph-Th Department, CERN, 1211 Geneva 23 Switzerland
and
Scuola Normale Superiore‡ and INFN, Piazza dei Cavalieri 7, I-56126 Pisa Italy

E-mail: `sagnotti@sns.it`

Abstract. The theory of freely-propagating massless higher spins is usually formulated via gauge fields and parameters subject to trace constraints. We summarize a proposal allowing to forego them by introducing only a pair of additional fields in the Lagrangians. In this setting, external currents satisfy usual Noether-like conservation laws, the field equations can be nicely related to those emerging from Open String Field Theory in the low-tension limit, and if the additional fields are eliminated without reintroducing the constraints a geometric, non-local description of the theory manifests itself.

Based on the talks presented by D.F. at the "First Meeting of Young Researchers of the European Superstring Theory Network", Heraklion, Crete, October 17-28 2005 and at the "Annual Meeting of the Italian String Theory Network", Pisa, November 4-5 2005, and on his Ph.D. Thesis, to be defended at Università Roma Tre on February 2006, and on the talk presented by A.S. at the "Fourth Meeting on Constrained Dynamics and Quantum Gravity", Cala Gonone (Sardinia, Italy), September 12-16 2005.

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† EU pre-doctoral Fellow. Address after October 1, 2006: Institute of Fundamental Physics, Chalmers University of Technology, 412 96 Goteborg Sweden.
‡ Permanent address after September 1, 2006.

1. Introduction

In four dimensions a massive spin-s boson can be described via a totally symmetric rank-s tensor $\varphi_{\mu_1\ldots\mu_s}$ subject to the conditions [1]

$$
\begin{aligned}
(\Box + m^2)\, \varphi_{\mu_1\ldots\mu_s} &= 0\,, \\
\partial^{\mu_1} \varphi_{\mu_1\ldots\mu_s} &= 0\,, \\
\varphi^{\nu}{}_{\nu\mu_1\ldots\mu_{s-2}} &= 0\,,
\end{aligned}
\tag{1}
$$

where the two constraints on the trace and the divergence remove the polarizations not within a single *irreducible* representation of the Poincaré Group. Their distinct role was clearly recognized long ago by Fierz and Pauli [2] when they tried to analyze the interaction with an electromagnetic field. While the first two equations in (1) became incompatible, the trace conditions proved harmless from the point of view of the minimal substitution, and they proposed to proceed in the search for an action principle leaving them as *algebraic constraints*, a choice that was to influence all subsequent research in the field. The Fierz-Pauli program of constructing an action principle leading to eqs. (1) was completed only in 1974 by Singh and Hagen [3], who also confined their attention to *massive* fields, while the massless limit of their Lagrangians was subsequently investigated by Fronsdal [4] and Fang and Fronsdal [5]. The resulting abelian gauge theories have been regarded since then as a paradigm for the free dynamics of higher-spin gauge fields.

In the Fronsdal equation for a spin-s boson,

$$
\mathcal{F}_{\mu_1\ldots\mu_s} \equiv \Box\varphi_{\mu_1\ldots\mu_s} - (\partial_{\mu_1}\partial^{\rho}\varphi_{\rho\mu_2\ldots\mu_s} + \ldots) + (\partial_{\mu_1}\partial_{\mu_2}\varphi^{\rho}{}_{\rho\mu_3\ldots\mu_s} + \ldots) = 0, \tag{2}
$$

where terms completing the symmetrizations are left implicit, the gauge field $\varphi_{\mu_1\ldots\mu_s}$ is a totally symmetric rank-s tensor subject to the condition that its double trace $\varphi^{\rho\sigma}{}_{\rho\sigma\mu_5\ldots\mu_s}$ *vanish identically*, while the parameter $\Lambda_{\mu_1\ldots\mu_{s-1}}$ entering the gauge transformation $\delta\,\varphi_{\mu_1\ldots\mu_s} = \partial_{\mu_1}\Lambda_{\mu_2\ldots\mu_s} + \ldots$, under which eq. (2) is invariant, is a *traceless* rank-$(s-1)$ tensor. Similar conditions are to be met for fermions, whose equations, for the corresponding cases of symmetric rank-n spinor-tensors $\psi_{\mu_1\ldots\mu_n}$, are

$$
\mathcal{S}_{\mu_1\ldots\mu_n} \equiv \slashed{\partial}\,\psi_{\mu_1\ldots\mu_n} - (\partial_{\mu_1}\,\slashed{\psi}_{\mu_2\ldots\mu_n} + \ldots) = 0\,. \tag{3}
$$

In the fermionic case the *triple γ-trace* of the gauge field is required to vanish, $\gamma^{\alpha}\gamma^{\beta}\gamma^{\gamma}\psi_{\alpha\beta\gamma\mu_4\ldots\mu_n} \equiv 0$, while the gauge parameter $\epsilon_{\mu_1\ldots\mu_{n-1}}$, entering the transformation law $\delta\,\psi_{\mu_1\ldots\mu_n} = \partial_{\mu_1}\epsilon_{\mu_2\ldots\mu_n} + \ldots$, is a symmetric rank-$(n-1)$ spinor-tensor whose γ-*trace* is required to vanish, $\gamma^{\rho}\epsilon_{\rho\mu_2\ldots\mu_{n-1}} \equiv 0$. This class of higher-spin gauge fields, while not exhaustive in more than four dimensions, is nonetheless a convenient ground for arriving at general results, that can typically be extended to cases of mixed symmetry without major difficulties. Here, following a common practice, in arbitrary dimensions we shall often call loosely "spin" the rank s of the bosonic gauge fields, or the rank n of the fermionic fields augmented by $\frac{1}{2}$.

These trace and γ−trace constraints, that as we have stressed are somehow a legacy of similar conditions on the massive fields of [2], appear somewhat unpleasing, and thus

we were motivated to reformulate the theory in such a way as to bypass them. Actually, a positive answer to this query had already been given by Pashnev, Tsulaia, Buchbinder and others in [6, 7], where the free dynamics of *individual and unconstrained* higher-spin gauge fields was first described via BRST techniques [8]. Their construction, however, requires many additional fields, their total number increasing linearly with the spin. This is to be contrasted with the approach resulting from our investigations [9, 10], that is being reviewed here, that removes the trace constraints from the Fronsdal and Fang-Fronsdal Lagrangians for all spins via *at most a pair of additional fields*.

A convenient starting point for our constructions are the Lagrangians related to eqs. (2) and (3), here written for *unconstrained* fields that transform with *unconstrained* gauge parameters. Whereas under these conditions the Lagrangians *are not* gauge invariant, in Section 2 we shall see that in the bosonic case a rank-$(s-3)$ compensator field $\alpha_{\mu_1\ldots\mu_{s-3}}$ transforming as $\delta\,\alpha_{\mu_1\ldots\mu_{s-3}} = \Lambda^{\rho}{}_{\rho\,\mu_1\ldots\mu_{s-3}}$, suffices to eliminate from the variation all terms involving the trace of the gauge parameter. Interestingly, the remainder is then proportional to the *gauge invariant combination*

$$\varphi^{\rho\,\sigma}{}_{\rho\,\sigma\,\mu_5\ldots\mu_s} - 4\,\partial\cdot\alpha_{\mu_5\ldots\mu_s} - \left(\partial_{\mu_5}\,\alpha^{\rho}{}_{\rho\mu_6\ldots\mu_s} + \ldots\right), \tag{4}$$

and as a result once this last constraint is enforced by a Lagrange multiplier β one ends up with *fully gauge invariant* Lagrangians, whose equations of motion, as reviewed in Section 3, propagate indeed the correct polarizations. Similar results hold for fermionic fields as well: in all cases (including actually the BRST formulations of [6, 7], as shown in [11]), the field equations can be reduced to the local non-Lagrangian compensator equations of [12, 11], that in the bosonic case read simply

$$\mathcal{F}_{\mu_1\ldots\mu_s} = 3\,\partial_{\mu_1}\partial_{\mu_2}\partial_{\mu_3}\,\alpha_{\mu_4\ldots\mu_s} + \ldots\,,$$
$$\varphi^{\rho\sigma}{}_{\rho\sigma\mu_5\ldots\mu_s} = 4\,\partial\cdot\alpha_{\mu_5\ldots\mu_s} + \left(\partial_{\mu_5}\alpha^{\rho}{}_{\rho\mu_6\ldots\mu_s} + \ldots\right), \tag{5}$$

where α denotes the rank-$(s-3)$ compensator defined above.

In Section 4 we shall review how the additional fields can be eliminated without reintroducing any trace constraints: the end result will be a *non-local* theory along the lines of [13, 12]. For instance, for spin three the non-local equations one obtains are

$$\mathcal{F}_{\mu_1\mu_2\mu_3} - \frac{1}{3\,\square}\left(\partial_{\mu_1}\partial_{\mu_2}\mathcal{F}^{\rho}{}_{\rho\mu_3} + \partial_{\mu_2}\partial_{\mu_3}\mathcal{F}^{\rho}{}_{\rho\mu_1} + \partial_{\mu_3}\partial_{\mu_1}\mathcal{F}^{\rho}{}_{\rho\mu_2}\right) = 0\,, \tag{6}$$

where $\mathcal{F}$ is defined in (2). The dynamical content of this theory is again the usual one of the Fronsdal formulation, as we shall see following [13, 12, 10].

Given the relative simplicity of the local formulation, the actual interest in the non-local equations is mostly driven by conceptual issues. Their most appealing feature is indeed a direct link to the *geometry* of these higher-spin fields, since the non-local equations obtained eliminating the additional fields confer a dynamical meaning to the higher-spin curvatures introduced in [14] by de Wit and Freedman, that do not play a direct role in the Fronsdal formulation. As will be explained in Section 5, these curvatures involve as many derivatives as the spin of the basic fields, and for this simple reason they can only enter two-derivative equations provided they are accompanied by

inverse powers of the d'Alembertian operator, but all resulting non-localities can be consistently eliminated by a partial gauge-fixing procedure involving the trace of the gauge parameter. The end result of this procedure was anticipated in [13], and has the suggestive feature of being a direct generalization of the geometric equations known for the more familiar spin-one and spin-two cases. Thus, if $\mathcal{R}_{\mu_1\mu_2\mu_3,\,\rho_1\rho_2\rho_3}$ is the curvature for the spin-three case, fully symmetric within the two sets of indices, it is possible to show that the non-local equation (6) can be written in the form

$$\frac{1}{\Box}\,\eta^{\rho_1\rho_2}\,\partial^\sigma\,\mathcal{R}_{\sigma\rho_1\rho_2,\,\mu_1\mu_2\mu_3} \;=\; 0\,, \tag{7}$$

where $\eta^{\rho_1\rho_2}$ denotes the flat Minkowski metric, a direct generalization of the spin-one Maxwell equation. In a similar fashion, for spin four, the first non-trivial even-spin case, the non-local geometric equations take the form [13, 12]

$$\frac{1}{\Box}\,\eta^{\rho_1\rho_2}\,\eta^{\rho_3\rho_4}\,\mathcal{R}_{\rho_1\rho_2\rho_3\rho_4,\,\mu_1\mu_2\mu_3\mu_4} \;=\; 0\,, \tag{8}$$

in clear analogy with the linearized spin-two Einstein equation. It should be appreciated that the spin-one and spin-two cases are the only ones for which these geometric equations are local.

The need for a better understanding of the relation between Higher-Spin Gauge Theory and String Theory is a serious motivation behind the interest in the former that is building up nowadays, and some elementary related issues will be reviewed in Section 6, where free equations emerging from String Field Theory [15] in the low-tension limit will be compared with the compensator equations (5). In that limit, as was known for a long time, the structure of the String equations simplifies to systems of three fields usually called "triplets" [16, 12, 17] (or their fermionic analogues of [12], or generalizations thereof [16, 11]). In all cases, the resulting fields are not subject to any trace constraints and display a correspondingly unconstrained gauge symmetry. A consistent truncation of the propagating modes then suffices to recover the local compensator equations (5) for spin-s bosons and corresponding ones for spin-$(n + \frac{1}{2})$ fermions. This result and its generalizations to the mixed-symmetry case [11] can be taken as a hint that the unconstrained formulation should be of help in unraveling the precise relationship between higher-spin systems and String theory at the interacting level. A precise statement to this effect, however, would still require further investigation, in particular since at present the only window on consistent higher-spin interactions is provided by the Vasiliev construction [18, 19, 20], which is strictly an on-shell formulation. We shall therefore conclude in Section 7 with some cursory remarks on the role of the unconstrained gauge symmetry in this construction, following [23]. The key result will be that the extended symmetry can indeed be accommodated in the free Vasiliev equations based on vector oscillators [19], that are actually closer in spirit to an off-shell form than their original version of [18] based on spinor oscillators. Interestingly, the compensators α do not appear explicitly in the equations. Rather, they emerge as exact forms of a generalized cohomological problem first discussed by Dubois-Violette and Henneaux [24].

In order to simplify the discussion, in the following we shall resort to a concise notation that we found quite useful in our previous works. Thus, primes (or bracketed suffixes) will denote traces, while all indices carried by the symmetric tensors $\varphi_{\mu_1 \dots \mu_s}$ and $\Lambda_{\mu_1 \dots \mu_{s-1}}$, by the metric tensor $\eta_{\mu\nu}$ or by derivatives will be left implicit. In addition, all terms will be understood as totally symmetrized, so that for instance $\partial \varphi$ will stand for $\partial_{\mu_1} \varphi_{\mu_2 \dots \mu_{s+1}} + \dots$. A few rules needed to take full advantage of this notation are:

$$
\begin{aligned}
(\partial^p \varphi)' &= \Box\, \partial^{p-2} \varphi + 2 \partial^{p-1} \partial \cdot \varphi + \partial^p \varphi', \\
\partial^p \partial^q &= \binom{p+q}{p} \partial^{p+q}, \\
\partial \cdot (\partial^p \varphi) &= \Box\, \partial^{p-1} \varphi + \partial^p \partial \cdot \varphi, \\
\partial \cdot \eta^k &= \partial\, \eta^{k-1}, \\
(\eta^k \varphi)' &= [D + 2(s+k-1)]\, \eta^{k-1} \varphi + \eta^k \varphi'.
\end{aligned}
\tag{9}
$$

For brevity, in the following Sections our discussion will be restricted to bosonic fields, although all results presented extend to fermionic fields, as discussed in [13, 12, 11, 9, 10]*.

2. "Minimal" unconstrained Lagrangians for higher-spin fields

Let us consider the Fronsdal Lagrangian [4]

$$
\mathcal{L}_0 = \frac{1}{2}\, \varphi \left(\mathcal{F} - \frac{1}{2}\, \eta\, \mathcal{F}' \right),
\tag{10}
$$

written for an *unconstrained* gauge field. Its *unconstrained* variation does not vanish, but

$$
\begin{aligned}
\delta \mathcal{L}_0 ={}& \Lambda' \binom{s}{3} \left\{ \frac{3}{4}\, \partial \cdot \mathcal{F}' - \frac{3}{2}\, \partial \cdot \partial \cdot \partial \cdot \varphi + \frac{9}{4}\, \Box\, \partial \cdot \varphi' \right\} \\
&- 9 \binom{s}{4} \partial \cdot \partial \cdot \varphi'\, \partial \cdot \Lambda' + \frac{15}{2} \binom{s}{5} \partial \cdot \partial \cdot \partial \cdot \varphi'\, \Lambda'' \\
&- 3 \binom{s}{4} \varphi''\, \partial \cdot \partial \cdot \partial \cdot \Lambda.
\end{aligned}
\tag{11}
$$

It is possible to add new terms involving the gauge field φ and a spin-$(s-3)$ "compensator" field α in such a way that all contributions involving the trace Λ' of the gauge parameter disappear in the resulting variation [9]. This procedure does not eliminate all unwanted terms. Rather, it generates a remainder proportional to the triple divergence of Λ,

$$
\delta \{ \mathcal{L}_0 + \mathcal{L}_1 + \mathcal{L}_2 \} = -3 \{ \varphi'' - 4 \partial \cdot \alpha - \partial \alpha' \} \partial \cdot \partial \cdot \partial \cdot \Lambda,
\tag{12}
$$

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* The web site http://www.ulb.ac.be/sciences/ptm/pmif/Solvay1proc.pdf contains the Proceedings of the First Solvay Workshop on Higher-Spin Gauge Fields, held in Brussels on May 12-14 2004, with a number of contributions, including [25, 26, 27, 28, 23] that are more closely related to this work, and many additional references to the original literature.

that however is proportional to the *fully gauge invariant* combination

$$\varphi'' - 4\,\partial\cdot\alpha - \partial\,\alpha'\,. \tag{13}$$

This analysis rests on the Bianchi identity for the Fronsdal operator,

$$\partial\cdot\mathcal{F} - \frac{1}{2}\,\partial\,\mathcal{F}' = -\frac{3}{2}\,\partial^3\,\varphi''\,, \tag{14}$$

that contains a classical "anomaly" for spin $s > 3$.

As we have recalled in the Introduction, the combination in (13) plays a role in the compensator equations (5) of [12, 11]. One can finally eliminate this variation introducing another field, β, that actually behaves as a Lagrange multiplier for second of (5) and transforms according to

$$\delta\beta = \partial\cdot\partial\cdot\partial\cdot\Lambda\,, \tag{15}$$

proportionally to the triple divergence of the gauge parameter. The resulting Lagrangian for an *unconstrained* spin-s boson [9],

$$\begin{aligned}
\mathcal{L} = {} & \frac{1}{2}\,\varphi\left(\mathcal{F} - \frac{1}{2}\eta\,\mathcal{F}'\right) - \binom{s}{3}\alpha\left\{\frac{3}{4}\partial\cdot\mathcal{F}' - \frac{3}{2}\partial\cdot\partial\cdot\partial\cdot\varphi + \frac{9}{4}\square\partial\cdot\varphi'\right\} \\
& + 9\binom{s}{4}\partial\cdot\alpha\,\partial\cdot\partial\cdot\varphi' - \frac{15}{2}\binom{s}{5}\alpha'\,\partial\cdot\partial\cdot\partial\cdot\varphi' + \frac{9}{4}\binom{s}{3}\alpha\,\square^2\,\alpha \\
& - 27\binom{s}{4}\partial\cdot\alpha\,\square\,\partial\cdot\alpha + 45\binom{s}{5}(\partial\cdot\partial\cdot\alpha)^2 + \frac{45}{2}\binom{s}{5}\partial\cdot\partial\cdot\alpha\,\square\,\alpha' \\
& - 45\binom{s}{6}\partial\cdot\partial\cdot\partial\cdot\alpha\,\partial\cdot\alpha' + 3\binom{s}{4}\beta\,(\varphi'' - 4\,\partial\cdot\alpha - \partial\,\alpha')\,,
\end{aligned} \tag{16}$$

is then invariant under the *unconstrained* gauge transformations

$$\begin{aligned}
\delta\varphi &= \partial\Lambda\,, \\
\delta\alpha &= \Lambda'\,, \\
\delta\beta &= \partial\cdot\partial\cdot\partial\cdot\Lambda\,.
\end{aligned} \tag{17}$$

There is an alternative, simpler, way to build these unconstrained Lagrangians. The starting point is in this case

$$\mathcal{L}_0 = \frac{1}{2}\,\varphi\left(\mathcal{A} - \frac{1}{2}\eta\,\mathcal{A}'\right)\,, \tag{18}$$

where the *fully* gauge invariant tensor $\mathcal{A}$, defined as

$$\mathcal{A} = \mathcal{F} - 3\,\partial^3\,\alpha\,, \tag{19}$$

enters the first of eqs. (5). As was the case for the Fronsdal operator $\mathcal{F}$ (see eq. (14)), its Bianchi identity contains a violation related to the double trace of the gauge field φ, that does not allow one to regard $\mathcal{A} - \eta/2\,\mathcal{A}'$ as a proper linearized Einstein-like tensor for higher spins. The key difference with respect to the Fronsdal operator, however, is that now the fully gauge invariant combination (13) is involved, so that

$$\partial\cdot\mathcal{A} - \frac{1}{2}\,\partial\,\mathcal{A}' = -\frac{3}{2}\,\partial^3(\varphi'' - 4\,\partial\cdot\alpha - \partial\,\alpha')\,. \tag{20}$$

The trial Lagrangian $\mathcal{L}_0$ has therefore a simple variation, that can be nicely expressed in terms of a pair of *gauge-invariant quantities*:

$$\delta \mathcal{L}_0 = \frac{3}{4}\binom{s}{3}\Lambda' \partial \cdot \mathcal{A}' + 3\binom{s}{4}\partial \cdot \partial \cdot \partial \cdot \Lambda \left(\varphi'' - 4\partial \cdot \alpha - \partial \alpha'\right). \tag{21}$$

In the same spirit as above, but in a simpler fashion, it is now possible to compensate this remainder introducing the fields α and β, and the complete Lagrangian is then simply

$$\mathcal{L} = \frac{1}{2}\varphi\left(\mathcal{A} - \frac{1}{2}\eta\,\mathcal{A}'\right) - \frac{3}{4}\frac{\alpha}{4}\binom{s}{3}\partial \cdot \mathcal{A}' - 3\,\beta\binom{s}{4}\left(\varphi'' - 4\partial \cdot \alpha - \partial \alpha'\right). \tag{22}$$

This alternative way of presenting (16) has the virtue of allowing handy generalizations to the cases of (A)dS backgrounds, that will presented elsewhere [29]. Similar results that hold for fermion fields are not discussed here for brevity, but can be found in [9, 10, 29].

3. Structure and meaning of the field equations

In the previous Section we have reviewed how invariant bosonic Lagrangians can be built introducing at most two additional fields. Differently from the Fronsdal fields, the gauge fields φ involved in this description are not subject to any trace constraints, and in order to check the consistency of the result one would thus like to display the dynamical content of eq. (16). To this end let us consider the field equations, and let us show that they can be reduced to the Fronsdal equations (2) by a partial gauge fixing.

The equations following from the Lagrangian (16) for the fields φ, β and α are of the form

$$\begin{aligned} \mathcal{A} - \frac{1}{2}\eta\,\mathcal{B} + \eta^2\,\mathcal{C} &= 0\,, \\ \varphi'' - 4\,\partial \cdot \alpha - \partial\,\alpha' &= 0\,, \\ \mathcal{G}_{\varphi,\beta}(\alpha) &= 0\,, \end{aligned} \tag{23}$$

with

$$\mathcal{A} \equiv \mathcal{F} - 3\,\partial^3\,\alpha\,, \tag{24}$$

where the explicit expressions for $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{G}$, not needed for the present discussion, can be found in [9]. It should be noted that, when β is on-shell, the first of (23) reduces to

$$\mathcal{A} - \frac{1}{2}\eta\,\mathcal{A}' + \eta^2\,\mathcal{C} = 0\,, \tag{25}$$

while the double trace of $\mathcal{A}$ vanishes *identically*. A straightforward analysis [9, 10] then shows that the equation for φ implies the independent relations $\mathcal{C} = 0$, $\mathcal{A}' = 0$, and finally the first of (5),

$$\mathcal{A} \equiv \mathcal{F} - 3\,\partial^3\,\alpha = 0\,. \tag{26}$$

Once this form is reached, the Fronsdal equation (2) can be directly recovered after a partial gauge fixing, making use of the trace Λ' of the gauge parameter, that can shift

away the field α, while the second of (23) removes at the same time the double trace of the gauge field φ.

This discussion completes the proof that the Fronsdal theory is gauge equivalent to the unconstrained formulation. Nonetheless, the unconstrained Lagrangians possess a wider gauge symmetry, and one may wonder whether this could play a role in the non-linear deformations of the theory.

The first hint that the wider gauge symmetry can imply some relevant differences with the Fronsdal theory as soon as one moves away from the free case can be appreciated comparing the properties of *external currents* coupled to the physical fields in the two approaches. Let us indeed introduce the coupling to an external source $\mathcal{J}$ in the standard fashion, adding to the Lagrangian (16) a $\varphi \cdot \mathcal{J}$ term. The equations of motion become in this case

$$
\begin{aligned}
&\mathcal{A} - \frac{1}{2}\eta\,\mathcal{B} + \eta^2\,\mathcal{C} = \mathcal{J}\,, \\
&\varphi'' - 4\,\partial \cdot \alpha - \partial\,\alpha' = 0\,, \\
&\mathcal{G}_{\varphi,\beta}(\alpha) = 0\,,
\end{aligned}
\tag{27}
$$

and it is possible to show that the divergence of the left-hand side of the first equation is proportional to the operator $\mathcal{G}$ defining the field equation of α [9, 10]. Explicitly, the divergence of the first equation in (27) gives $\frac{\eta}{4}\,\mathcal{G}_{\varphi,\beta}(\alpha) = \partial \cdot \mathcal{J}$, and therefore, because of the third of (27), *the source must be divergence-free on-shell*, as expected from Noether's theorem. This result shows that the properties of the unconstrained system are consistent with physical expectations for the most natural interactions in a Field Theory, and marks a sizeable difference with the Fronsdal scheme. In this last case, in fact, the corresponding equations are

$$
\mathcal{F} - \frac{1}{2}\,\eta\,\mathcal{F}' = \mathcal{J}\,,
\tag{28}
$$

and consequently their divergence yields $-\frac{1}{2}\eta\,\partial \cdot \mathcal{F}' = \partial \cdot \mathcal{J}$. Hence, while for $s = 1, 2$ the usual condition $\partial \cdot \mathcal{J} = 0$ is recovered, in Fronsdal's theory from spin $s = 3$ onwards consistency only implies the weaker condition that *the traceless part of the divergence* vanish. Although sufficient to guarantee that only physical polarizations contribute to the exchange of quanta between sources [4], this does not completely fit the conditions for a Noether current. An equivalent argument can be given for fermions, as in [9, 10].

4. Removing the additional fields: non-local formulation

In this Section we would like to show that it is possible to eliminate the auxiliary fields from the unconstrained theory to recover a formulation which is closer in spirit to the familiar cases of spin one or two [12, 10]. The resulting dynamics is not local, but it is again possible to reduce it to the local Fronsdal form by a partial gauge fixing involving the trace of the gauge parameter.

The idea is to look for minimal combinations of the $\mathcal{F}$ operator with its traces and divergences, not involving α and such that the resulting expression vanishes identically

when the first of (5) is satisfied. In the spin-3 case one can find in this fashion two independent, fully gauge invariant, unconstrained equations of this type,

$$\mathcal{F} - \frac{1}{3}\frac{\partial^2}{\Box}\mathcal{F}' = 0, \tag{29}$$

$$\mathcal{F} - \frac{\partial^3}{\Box^2}\partial\cdot\mathcal{F}' = 0. \tag{30}$$

The second can be obtained combining the first with its trace, but is anyway quite interesting, since it is of the form

$$\mathcal{F} = 3\,\partial^3\,\mathcal{H}(\varphi), \tag{31}$$

where $\mathcal{H}(\varphi) = \frac{1}{3\Box^2}\partial\cdot\mathcal{F}'$ is such that $\delta\,\mathcal{H}(\varphi) = \Lambda'$. In other words, the non-local construct $\mathcal{H}$ plays the role of the compensator α!

We have thus seen that in the spin-3 case it is possible to eliminate α from the compensator equation arriving at a fully gauge invariant equation for the single field φ, without reintroducing any trace constraints. The irreducible equation is not local, as anticipated, but can be cast into the particularly useful form (31), in which it is manifest that using only of the trace Λ' of the gauge parameter *all non localities can be removed*, thus recovering the Fronsdal form $\mathcal{F} = 0$.

For bosons of arbitrary spin s one can reproduce the scheme just described for spin 3: starting from the first of (5), it is possible to find a linear combination of constructs built from $\mathcal{F}$ that vanishes identically and does not involve α, without reintroducing any trace constraints [13, 12]. To this end, let us define $\mathcal{F}^{(1)} \equiv \mathcal{F}$ and let us consider the sequence of kinetic operators

$$\mathcal{F}^{(n+1)} = \mathcal{F}^{(n)} + \frac{1}{(n+1)(2n+1)}\frac{\partial^2}{\Box}\mathcal{F}^{(n)\prime} - \frac{1}{n+1}\frac{\partial}{\Box}\partial\cdot\mathcal{F}^{(n)}. \tag{32}$$

They possess the crucial property that we need, since eqs. (5) and (32) imply that

$$\delta\,\mathcal{F}^{(n)} = (2n+1)\frac{\partial^{2n+1}}{\Box^{n-1}}\alpha^{[n-1]}. \tag{33}$$

Hence, for any spin s, if p denotes the integer part of $\{\frac{s-3}{2}\}$, the trace $\alpha^{[p+1]}$ of the compensator is simply not available, and hence the non-local kinetic operator $\mathcal{F}^{(p+2)}$ satisfies the irreducible gauge-invariant equation

$$\mathcal{F}^{(p+2)} = 0, \tag{34}$$

where the compensator α has disappeared. Moreover, these kinetic operators satisfy the modified "Bianchi identities" [13, 12]

$$\partial\cdot\mathcal{F}^{(n)} - \frac{1}{2n}\partial\,\mathcal{F}^{(n)\prime} = -\left(1+\frac{1}{2n}\right)\frac{\partial^{2n+1}}{\Box^{n-1}}\varphi^{[n+1]}. \tag{35}$$

where for n sufficiently large the classical "anomaly" also disappears, so that they eventually suffice to define for all spin-s fields fully gauge invariant analogues of the Einstein tensor,

$$\mathcal{G}^{(n)} = \sum_{p\leq n}\frac{(-1)^p}{2^p\,p!\,\binom{n}{p}}\,\eta^p\,\mathcal{F}^{(n)\,[p]}, \tag{36}$$

that have vanishing divergences like their spin-2 counterpart. From $\mathcal{G}^{(n)}$ one can then build generalized Lagrangians that are fully gauge invariant without any restrictions on gauge fields or gauge parameters. Making use repeatedly of eq. (35), it is also possible to show that eqs. (34) can always be reduced to the compensator-like form (31) and then finally gauge fixed to the local Fronsdal form [10]. Again, similar results hold for fermion fields, but are not discussed here for brevity and can be found in [13, 12, 10].

5. Higher-spin geometry from the non-local formulation

The non-local formulation, although dynamically equivalent to the local one, is clearly less manageable, both at the classical and at the quantum level. So, what advantages can it possibly have? Briefly stated, as we shall review in this Section, *the non-local formulation encodes a geometric description, for all higher spins, along lines familiar from the two cases of spin-one and spin-two fields.*

In order to motivate and explain this statement, our starting point will be provided by the hierarchy of "connections" introduced by de Wit and Freedman in [14], that again can be nicely motivated by a closer look at the first non-trivial case of a spin-3 boson. In analogy with the two lower-spin cases, the key idea is to consider a linear combination of first derivatives of the basic field $\varphi_{\alpha\beta\gamma}$ such that the resulting gauge transformation be as simple as possible. The proper choice is

$$\Gamma^{(1)}_{\rho,\alpha\beta\gamma} = \partial_\rho \varphi_{\alpha\beta\gamma} - (\partial_\alpha \varphi_{\rho\beta\gamma} + \partial_\beta \varphi_{\rho\alpha\gamma} + \partial_\gamma \varphi_{\alpha\beta\rho}) , \tag{37}$$

since indeed

$$\delta\Gamma^{(1)}_{\rho,\alpha\beta\gamma} = -2(\partial_\alpha\partial_\beta\Lambda_{\rho\gamma} + \partial_\alpha\partial_\gamma\Lambda_{\rho\beta} + \partial_\beta\partial_\gamma\Lambda_{\rho\alpha}) , \tag{38}$$

where for the sake of clarity all indices have been displayed. It is possible to simplify further the gauge transformation taking a combination of *second* derivatives of φ: the proper choice is now

$$\Gamma^{(2)}_{\rho\sigma,\alpha\beta\gamma} = \partial_\rho\Gamma^{(1)}_{\sigma,\alpha\beta\gamma} - \frac{1}{2}(\partial_\alpha\Gamma^{(1)}_{\sigma,\rho\beta\gamma} + \dots) , \tag{39}$$

since the resulting variation is simply

$$\delta\,\Gamma^{(2)}_{\rho\sigma,\alpha\beta\gamma} = 3\,\partial_\alpha\partial_\beta\partial_\gamma\Lambda_{\rho\sigma} , \tag{40}$$

that resembles more closely the behavior of the spin-2 Christoffel connection or of the spin-1 gauge field. In this respect, for spin 3 this $\Gamma^{(2)}$, more than $\Gamma^{(1)}$, is what one would properly call a "connection", and indeed a sort of "symmetric curl" of $\Gamma^{(2)}$ builds a gauge invariant "curvature" for the spin 3 field φ:

$$\Gamma^{(3)}_{\rho\sigma\tau,\,\alpha\beta\gamma} \equiv \mathcal{R}^{(3)}_{\rho\sigma\tau,\,\alpha\beta\gamma} = \partial_\rho\Gamma^{(2)}_{\sigma\tau,\,\alpha\beta\gamma} - \frac{1}{3}(\partial_\alpha\Gamma^{(2)}_{\sigma\tau,\,\rho\beta\gamma} + \dots) . \tag{41}$$

In order to deal with the general case, one can follow a strategy that should be clear from the previous example, the key observation being that in each $\Gamma^{(m)}_{\rho_1\dots\rho_m,\,\alpha_1\dots\alpha_s}$ the two groups of indices *are not equivalent*. The ρ's are "special" indices, related to the introduction of m derivatives at the m-th step, and the combinations should be chosen

in such a way that in the gauge variation of $\Gamma^{(m)}$ the special indices end within the gauge parameter $\Lambda_{\mu_1 \ldots \mu_{s-1}}$. If this is attained, in the gauge transformation of $\Gamma^{(s-1)}$ *all* indices belonging to the gauge parameter will be special ones, and consequently $\Gamma^{(s)}$ (along with any $\Gamma^{(s+k)}$ for $k > 0$) will be necessarily gauge invariant, simply because there will be no room in Λ to accommodate more special indices. In order to present these results in a compact fashion, let us resort to a mixed notation, in which the "∂" symbol is reserved for derivatives with respect to "special" indices while "∇" is taken to denote derivatives with respect to the remaining ones. Bearing in mind that symmetrization only applies to pairs of indices within each of the two sets, one could write the de Wit-Freedman symbols in the form

$$\Gamma^{(m)} = \sum_{k=0}^{m} \frac{(-1)^k}{\binom{m}{k}} \, \partial^{m-k} \, \nabla^k \, \varphi \,, \tag{42}$$

and the corresponding gauge transformations,

$$\delta \, \Gamma^{(m)} = (-1)^m \, (m+1) \, \nabla^{m+1} \, \Lambda \,, \tag{43}$$

then display their direct correspondence with the kinetic operators $\mathcal{F}^{(n)}$ of the previous Section (see eq. (33). In particular, in this notation the gauge-invariant curvatures become

$$\mathcal{R}^{(s)} = \sum_{k=0}^{s} \frac{(-1)^k}{\binom{s}{k}} \, \partial^{s-k} \, \nabla^k \, \varphi \,. \tag{44}$$

These curvatures *can not* be used directly to describe the Fronsdal dynamics, simply because they are higher-derivative objects, and this was the reason why the authors of [14] rejected the possibility of describing higher-spin fields in an *unconstrained* fashion, returning at the end of their analysis to the constrained Fronsdal formulation. And indeed, if one assumes that $\Lambda' \equiv 0$, the trace of each connection in the hierarchy is also gauge invariant, and in particular the trace of the $\Gamma^{(2)}$ defines a gauge invariant second order operator, that coincides with $\mathcal{F}$ and in [14] was chosen to define the field equation, thus justifying Fronsdal's result from a different viewpoint.

Nonetheless, one can well try and describe via the curvatures (44) an "effective" second-order dynamics, acting on them with suitable inverse powers of the d'Alembertian operator in order to recover the physical dimensions of a second-order operator. The resulting *geometric* equations [13, 12]

$$\frac{1}{\Box^{n-1}} \, \mathcal{R}^{[n]}{}_{\mu_1 \cdots \mu_{2n}} = 0 \,, \qquad (s = 2\,n) \,, \tag{45}$$

$$\frac{1}{\Box^{n}} \, \partial \cdot \mathcal{R}^{[n-1]}{}_{\mu_1 \cdots \mu_{2n-1}} = 0 \,, \quad (s = 2\,n+1) \,, \tag{46}$$

are a nice way of encoding the results of the previous Section, once the curvatures are *identified* with suitably iterated $\mathcal{F}^{(n)}$ operators. These geometric equations are natural generalizations of the well-known spin-1 and spin-2 cases, are of course non-local on account of our preceding arguments, but as we have reviewed in previous Sections the non local terms are harmless gauge artifacts, as was the case for the non-local terms in

the equivalent kinetic operators (32). These equations were generalized to the case of mixed symmetry in [30, 31].

In conclusion, *the curvatures of de Wit and Freedman acquire a direct dynamical meaning in the non-local theory*, whose equivalence to the local unconstrained one has been discussed in the previous Section. This description embodies the linearized forms of the non-linear Yang-Mills and Einstein equations, for spin 1 and 2 respectively, and leads naturally to speculate that a currently unknown non-linear metric-like geometry (to be contrasted with Vasiliev's frame-like geometry [18, 19, 20]) could alternatively be taken to underlie higher-spin interactions, in such a way as to give rise to eqs. (45) and (46) in the linearized limit. Similar results hold for fermionic fields, and are discussed in some detail in [10].

6. Higher-spin geometry and String Field Theory

Higher-spin states are an intrinsic, inevitable part of the string spectrum. Hence, it is natural to look for links between these properties of higher-spin fields and what can be deduced from String Field Theory in the linearized approximation.

In this Section we would thus like to review briefly how the unconstrained formulation of free higher-spin gauge fields can be related to the low-tension limit of free String Field Theory. We shall depart slightly from [12, 11], and follow [10]. As shown in [15], in all cases the free equations of String Field Theory can be written in the form

$$\mathcal{Q} \, |\Phi\rangle \; = \; 0 \, , \tag{47}$$

where $\mathcal{Q}$ is the BRST operator of the first-quantized string. These systems display the chain of gauge invariances

$$\delta|\Phi^{(n)}\rangle \; = \; \mathcal{Q} \, |\Phi^{(n+1)}\rangle \, , \tag{48}$$

with $\Phi^{(1)}$ the string gauge field and $\Phi^{(n)}$ ($n > 1$) a corresponding chain of gauge parameters.

In the usual case of tensile strings, these equations describe *massive* higher-spin modes, with masses determined by the string tension, but it is important to stress that *neither* (47) *nor* (48) *involve trace conditions*. Hence, it is reasonable to expect that in the low-tension limit $\alpha' \to \infty$ the resulting massless dynamics should naturally relate to the unconstrained formulation reviewed in the previous Sections, rather than to Fronsdal's constrained equations.

Even for the open bosonic string, the complete analysis of the spectrum would require to deal with mixed-symmetry tensors, as discussed in [11]. Here it will suffice to note that for symmetric tensors (corresponding to states in the leading Regge trajectory of the open bosonic string, generated by the lowest string oscillators α_{-1}) the limit $\alpha' \to \infty$ yields *triplet* systems of the type [16, 12, 17]

$$\begin{aligned} \Box \, \phi &= \partial \, C \, , \\ \partial \cdot \phi - \partial \, D &= C \, , \\ \Box \, D &= \partial \cdot C \, , \end{aligned} \tag{49}$$

where ϕ, C and D are symmetric tensors of ranks s, $s-1$ and $s-2$ respectively. Notice that *no trace constraints are enforced on these fields*, while these equations are invariant under the *unconstrained* gauge transformations

$$
\begin{aligned}
\delta\phi &= \partial\Lambda\,, \\
\delta C &= \square\Lambda\,, \\
\delta D &= \partial\cdot\Lambda\,.
\end{aligned}
\tag{50}
$$

In order to establish contact with the unconstrained equations of the preceding Sections, one can proceed as in [13], or alternatively one can manipulate eq. (49) in order to reproduce the Fronsdal operator (2), as in [10]. In particular, making use of the second of (49) to express the gradient of $\partial\cdot\phi$ in terms of C and D according to

$$
\partial\partial\cdot\phi = 2\partial^2 D + \partial C\,,
\tag{51}
$$

and taking a double gradient of the trace of the first, after dividing by the d'Alembertian operator one can arrive at

$$
\partial^2\phi' = 2\partial^2 D + 3\frac{\partial^3}{\square}C'\,.
\tag{52}
$$

Putting together (51) and (52) with the first of (49), the final result,

$$
\mathcal{F} = 3\frac{\partial^3}{\square}C'\,,
\tag{53}
$$

is clearly of the form (5), with the identification of α with $\frac{1}{\square}C'$. And indeed, starting from the third of the triplet equations one can also arrive at

$$
\phi'' = 4\frac{1}{\square}\partial\cdot C' + \frac{\partial}{\square}C''\,.
\tag{54}
$$

Notice that in this fashion one has truncated all modes that are responsible for the propagation of lower spins $(s-2)$, $(s-4)$, ... in the triplet, so that the end result is to exhibit a direct link between the (truncated) triplet (49) and the compensator equations (5). To reiterate, in the low-tension limit the equations of String Field Theory for fully symmetric tensors bear a direct relationship to the unconstrained formalism for higher spins reviewed in Section 2 of this paper, and hence to the non-local geometric equations reviewed in Section 3. Similar results hold for mixed-symmetry tensors [11], and for fermion fields [12, 11, 10], but are not discussed here for the sake of brevity.

7. Unconstrained gauge symmetry and the Vasiliev equations

At present not much is known in a systematic fashion about higher-spin interactions, aside from the fact that they are bound to involve infinitely many fields, but a remarkable example is available. *Consistent non-linear interactions* for an infinite set of fully symmetric tensors can be elegantly encoded in the Vasiliev equations [18, 19, 20], a set of curvature constraints relating a one-form master field A containing the higher-spin fields and a zero-form master field Φ subsuming the content of corresponding Weyl curvatures (as well as an independent scalar mode). These fields may also be matrix-valued, in

analogy with standard Chan-Paton [21] constructions, which suggests a natural link with the leading Regge trajectory of open bosonic strings. The key idea behind these equations is to bypass the standard S-matrix no-go theorems for higher-spin interactions by introducing a cosmological term [22], that builds naturally an expansion about dS or AdS backgrounds rather than around flat space, and to generalize the frame formulation of gravity to higher spins by properly extending the $SO(1, D)$ ($SO(2, D − 1)$) tangent-space Lorentz algebra of $(A)dS_D$ Einstein gravity. Thus, while the latter can be realized via quadratic expressions in oscillators, the Vasiliev formulation rests on the use of arbitrary polynomials, that essentially define what are commonly called "higher-spin algebras".

There are actually two distinct versions of the Vasiliev equations. The original one [18] applies to the four-dimensional case, rests on the properties of Grassman-even oscillators ξ_α and $\xi_{\dot\alpha}$ that are spinors of the tangent-space Lorentz group, and if truncated results in free equations that are precisely in the constrained Fronsdal form, although they are partly presented in spinor notation. On the other hand, a more recent version [19] rests on a different set of Grassman-even oscillators, Y_A^i, that are simultaneously Lorentz vectors and doublets of an internal $Sp(2, R)$. It is closer in spirit to an off-shell form and, interestingly, can accommodate the extended symmetry reviewed in this paper. The internal $Sp(2, R)$ symmetry brings about some subtleties, requires suitable projections, and allows two distinct options. The first, usually called "weak projection", pursued by Vasiliev in [19] and reviewed in detail in [20], treats in a similar fashion the one-form A and the zero-form Φ, and at the free level reduces again to the Fronsdal formulation. The second option, usually called "strong projection" [23], restricts *only the zero form* Φ while letting the gauge fields adjust correspondingly.

Once trace constraints are not enforced on the gauge fields, it is natural to expect that the compensators α should play a role, but this happens in a rather amusing and surprising fashion. In the following we shall try to give a flavor of this result, following [23]. We should stress, however, that the prospects for the "strong" projection at the interacting level are not fully clear at the moment, since it apparently brings about Weyl-ordering divergences in non-linear terms, but on the other hand the off-shell origin of the Vasiliev equations, where the unconstrained formulation would be expected to play a key role, is also unclear. The recent work in [23] contains a preliminary analysis of the problem. Although with the "strong" projection divergences would naively manifest themselves in non-linear interactions, one can not exclude at the present time that *divergent field redefinitions* or other subtleties may eventually suffice to remove them[*]. A proper discussion of the Vasiliev equations and of all these issues, however, is beyond the scope of this review, and therefore we shall content ourselves with providing some hints.

The key fact is that, in building master fields with the vector oscillators Y_A^i that

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[*] A.S. is grateful to C. Iazeolla and P. Sundell for extensive discussions on this point.

satisfy the commutation relations

$$\left[Y_A^i, Y_B^j \right] \;=\; i\,\epsilon^{ij}\,\eta_{AB} \quad (A, B = 1, ..., D+1) \quad (i, j = 1, 2) \tag{55}$$

one ends up in principle with Taylor coefficients, the ordinary fields, that aside from filling the desired $SO(2, D-1)$ or $SO(1, D)$ representations generally transform under $Sp(2, R)$ as well.

The $Sp(2, R)$ internal symmetry, on the other hand, reflects a redundancy, its origin being the need to dispose of dual sets of "coordinates" (the Y_A^1) and "momenta" (the Y_A^2) in order to write eq. (55). Hence the need for a restriction to fields that are $Sp(2, R)$ singlets, that can be effected by suitable constraints on A and Φ, but in order to arrive at dynamical equations one is to go further, removing also traces from the Weyl tensors in Φ, for a reason that we shall now try to explain. To this end, following [23], and abiding to a similar notation, let us begin by recalling that, if all interactions are neglected and the cosmological constant is subsequently contracted away, the Vasiliev equations with a "strong" projection reduce to the flat-space relations ($s \geq 2; k = 0, \ldots, s-1$) [32]

$$\partial_{[\mu} W^{(s-1,k)}_{\nu],a(s-1),b(k)} \;+\; W^{(s-1,k+1)}_{[\mu|,a(s-1),|\nu]b(k)} \;=\; \delta_{k,s-1}\, C_{[\mu|a(s-1),|\nu]b(s-1)} \,. \tag{56}$$

Here $W^{(s-1,k)}$ identifies a field transforming according to a traceful two-row Young tableau, with $(s-1)$ boxes in the first row and k boxes in the second, in a convention where total row symmetrization is manifest, while for instance $a(s-1)$ and $b(k)$ denote two sets of $(s-1)$ and k fully symmetrized tangent-space vector indices. Eqs. (56) are invariant under the gauge transformations

$$\delta W^{(s-1,k)}_{\mu,a(s-1),b(k)} \;=\; -\partial_\mu\, \xi^{(s-1,k)}_{a(s-1),b(k)} \;+\; \xi^{(s-1,k+1)}_{a(s-1),b(k)\mu} \quad (k = 0, \ldots, s-2)$$
$$\delta W^{(s-1,s-1)}_{\mu,a(s-1),b(s-1)} \;=\; -\partial_\mu\, \xi^{(s-1,s-1)}_{a(s-1),b(s-1)} \,, \tag{57}$$

involving the sequence of gauge parameters $\xi^{(s-1,k)}_{a(s-1),b(k)}$, for $k = 0, \ldots, s-1$. It takes some work to bring the compact-looking non-linear equations of [19],

$$\widehat{F} \;=\; \frac{i}{2}\, dZ^i \wedge dZ_i\, \widehat{\Phi} \star \kappa \,, \qquad \widehat{D}\,\widehat{\Phi} \;=\; 0 \,, \tag{58}$$

to this form, since the former involve additional non-commutative Z coordinates to be disposed of via a perturbative expansion in powers of Φ. A nice discussion of how this can be achieved for the simpler spinorial construction can be found in [33]. At any rate, eq. (56) has a rather transparent meaning: it equates Riemann-like tensors to Weyl-like ones. If the latter are trace-free, the end result is then enforcing, in an indirect way nicely compatible with gauge invariance, via the trace of eq. (56), higher-spin generalizations of the Einstein equation "$Ricci = 0$".

Insofar as the free limit is concerned, working with the "strong" projection of [23] amounts to allowing in eqs. (56) one-forms W subject to no trace conditions while restricting the corresponding Weyl tensors in C to be traceless. Let us consider eq. (56) for the two cases of $s = 2$ and $s = 3$, in order to display explicitly what subtlety presents itself in the latter, leading eventually to the emergence of the compensator α.

The $s = 2$ case is the familiar example of linearized Einstein gravity in the frame formulation. Eqs. (56) becomes in this case a set of two conditions: the first,

corresponding to $k = 0$, is the "vielbein" postulate, that relates the spin connection $W^{(1,1)}_{\mu,a,b}$ to the vielbein $W^{(1,0)}_{\mu,a}$, while the trace of the second, corresponding to $k = 1$, gives the Einstein equation. A partial gauge fixing of $W^{(1,0)}_{\mu,a}$ to a symmetric $h_{\mu\nu}$ recovers the linearized Einstein equation in the metric form, that is also the $s = 2$ case of the Fronsdal equation (2). Notice that the condition that this gauge choice be preserved correlates the two gauge parameters ξ_a and $\xi_{a,b}$, in the $(1,0)$ and $(1,1)$, in such a way that

$$\xi_{a,b} = \frac{1}{2} \left(\partial_b \xi_a - \partial_a \xi_b \right), \tag{59}$$

which turns the gauge transformation of the vielbein

$$\delta W^{(1,0)}_{\mu a} = -\partial_\mu \xi_a + \xi_{a,\mu} \tag{60}$$

into a proper transformation for $h_{\mu\nu}$.

The novelty first shows up for spin $s = 3$, since in this case the iterative solution of the homogeneous constraints builds up at the second step a second-order operator $W^{(2,2)}_\mu$, whose trace in a pair of tangent-space indices has the right structure to be the field equation. This is fully determined by its original gauge transformation into the gradient of a parameter $\xi_{ab,cd}$ transforming in the $(2,2)$ of the tangent-space group, built from the lowest gauge parameter ξ_{ab} and two derivatives,

$$\xi^c_{ab,c} \sim \Box \xi_{ab} - \partial_a \partial \cdot \xi_b - \partial_b \partial \cdot \xi_a + \partial_a \partial_b \xi', \tag{61}$$

so that

$$\eta^{cd} W^{(2,2)}_{ab,cd} = \Box \varphi_{\mu ab} - \partial_a \partial \cdot \varphi_{\mu b} - \partial_b \partial \cdot \varphi_{\mu a} + \partial_a \partial_b \varphi'_\mu, \tag{62}$$

after gauge fixing $W^{(2,0)}_{\mu,ab}$ to a fully symmetric tensor $\varphi_{\mu\nu\rho}$.

However, given the trace-free condition on the Weyl tensor C, the $k = 2$ condition implies the additional constraint

$$\partial_{[\mu} W^{(2,2)}_{\nu],ab,c}{}^c = 0, \tag{63}$$

and therefore the term in eq. (62) is to be *a pure gradient*,

$$\Box \varphi_{ab\mu} - \partial_a \partial \cdot \varphi_{b\mu} - \partial_b \partial \cdot \varphi_{a\mu} + \partial_a \partial_b \varphi'_\mu = \partial_\mu \beta_{ab}, \tag{64}$$

but a curl of this implies an additional constraint on β:

$$\partial_{[\mu}(\partial \cdot \varphi_{a]b} - \partial_{a]}\varphi'_b) = \partial_{[\mu}\beta_{b]a}, \tag{65}$$

This constraint has an inhomogeneous solution,

$$\beta_{ab} = \left(\partial \cdot \varphi_{ab} - \partial_a \varphi'_b - \partial_b \varphi'_a \right), \tag{66}$$

but also *a homogenous solution*, and this is where the compensator α actually sits, as an "exact" form in the sense of Dubois-Violette and Henneaux [24].

Let us pause to see explicitly the subtlety. They key point is that, if one tries to solve the condition

$$\partial_{[\mu}\beta_{a]b} = 0, \tag{67}$$

the symmetry in (a,b) *does not allow* a single gradient. This is simple to see since, letting

$$\beta_{ab} = \partial_a \alpha_b + \partial_b \alpha_a \,, \tag{68}$$

eq. (67) implies indeed the additional constraint

$$\partial_b \partial_{[\mu} \alpha_{a]} = 0 \,, \tag{69}$$

whose solution is

$$\alpha_a = \partial_a \alpha \,. \tag{70}$$

This identifies the compensator α, and putting these results together one finally recovers the compensator equation (5) for $s = 3$. For higher spins one must similarly work backwards from the highest constraint to the two-derivative operator, *retaining the exact terms* emerging at subsequent steps and subjecting them to due constraints. The end result is in the general the recovery of eq. (5), with spin-$(s-3)$ compensators emerging from terms that are exact in the sense of [24]. This derivation resonates with a result found by Bekaert and Boulanger [30] in their analysis of the link between the compensator α and the hierarchy of Freedman-deWit connections of [14], but it is actually different since, despite some unfortunate nomenclature in [23], in the Vasiliev construction one is working with generalized spin connections, rather than with generalized Christoffel symbols.

Acknowledgements

[1] M. Fierz, Helv. Phys. Acta **12** (1939) 3.
[2] M. Fierz and W. Pauli, Proc. Roy. Soc. Lond. **A173** (1939) 211-232.
[3] L. P. Singh and C. R. Hagen, Phys. Rev. D9 (1974) 898, Phys. Rev. D9 (1974) 910.
[4] C. Fronsdal, Phys. Rev. D **18** (1978) 3624.
[5] J. Fang and C. Fronsdal, Phys. Rev. D **18** (1978) 3630.
[6] A. Pashnev and M. M. Tsulaia, Mod. Phys. Lett. A **12** (1997) 861 [arXiv:hep-th/9703010], Mod. Phys. Lett. A **13** (1998) 1853 [arXiv:hep-th/9803207]; C. Burdik, A. Pashnev and M. Tsulaia, Nucl. Phys. Proc. Suppl. **102** (2001) 285 [arXiv:hep-th/0103143]; I. L. Buchbinder, A. Pashnev and M. Tsulaia, Phys. Lett. B **523** (2001) 338 [arXiv:hep-th/0109067], arXiv:hep-th/0206026.
[7] I. L. Buchbinder, V. A. Krykhtin and A. Pashnev, Nucl. Phys. B **711** (2005) 367 [arXiv:hep-th/0410215].

[8] C. Becchi, A. Rouet and R. Stora, Commun. Math. Phys. **42** (1975) 127, Annals Phys. **98** (1976) 287; I.V. Tyutin, preprint FIAN n. 39 (1975); E. S. Fradkin and G. A. Vilkovisky, Phys. Lett. B **55** (1975) 224; I. A. Batalin and G. A. Vilkovisky, Phys. Lett. B **69** (1977) 309; M. Henneaux, Phys. Rept. **126** (1985) 1.

[9] D. Francia and A. Sagnotti, Phys. Lett. B **624** (2005) 93-104 [arXiv:hep-th/0507144].

[10] D. Francia, "Free geometric theory for higher-spin fields," Ph.D. Thesis, to appear.

[11] A. Sagnotti and M. Tsulaia, Nucl. Phys. B **682** (2004) 83 [arXiv:hep-th/0311257].

[12] D. Francia and A. Sagnotti, Class. Quant. Grav. **20** (2003) S473 [arXiv:hep-th/0212185].

[13] D. Francia and A. Sagnotti, Phys. Lett. B **543** (2002) 303 [arXiv:hep-th/0207002].

[14] B. de Wit and D. Z. Freedman, Phys. Rev. D **21** (1980) 358; T. Damour and S. Deser, Ann. Inst. H. Poincaré Phys. Theor. **47** (1987) 277.

[15] W. Siegel, Nucl. Phys. B **263** (1986) 93; W. Siegel and B. Zwiebach, Nucl. Phys. B **263** (1986) 105; T. Banks and M. E. Peskin, Nucl. Phys. B **264** (1986) 513; M. Kato and K. Ogawa, Nucl. Phys. B **212** (1983) 443; N. Ohta, Phys. Rev. D **33** (1986) 1681, Phys. Lett. B **179** (1986) 347; A. Neveu, H. Nicolai and P. C. West, Nucl. Phys. B **264** (1986) 573; A. Neveu and P. C. West, Nucl. Phys. B **268** (1986) 125; E. Witten, Nucl. Phys. B **268** (1986) 253. See also: N. Marcus and A. Sagnotti, Phys. Lett. B **178** (1986) 343.

[16] A. K. Bengtsson, Phys. Lett. B **182** (1986) 321; M. Henneaux and C. Teitelboim, in "Quantum Mechanics of Fundamental Systems, 2", eds. C. Teitelboim and J. Zanelli (Plenum Press, New York, 1988), p. 113.

[17] G. Bonelli, Nucl. Phys. B **669** (2003) 159 [arXiv:hep-th/0305155], JHEP **0311** (2003) 028 [arXiv:hep-th/0309222].

[18] M. A. Vasiliev, Phys. Lett. B243 (1990) 378, Phys. Lett. B285 (1992) 225.

[19] M. A. Vasiliev, Phys. Lett. B **567** (2003) 139 [arXiv:hep-th/0304049].

[20] For a recent review see:
X. Bekaert, S. Cnockaert, C. Iazeolla and M. A. Vasiliev, arXiv:hep-th/0503128.

[21] J. E. Paton and H. M. Chan, Nucl. Phys. B **10** (1969) 516; J. H. Schwarz, CALT-68-906-REV *Presented at 6th Johns Hopkins Workshop on Current Problems in High-Energy Particle Theory, Florence, Italy, Jun 2-4, 1982*; N. Marcus and A. Sagnotti, Phys. Lett. B **119** (1982) 97, Phys. Lett. B **188** (1987) 58. For reviews, see: J. H. Schwarz, Phys. Rept. **89** (1982) 223; C. Angelantonj and A. Sagnotti, Phys. Rept. **371** (2002) 1 [Erratum-ibid. **376** (2003) 339] [arXiv:hep-th/0204089].

[22] E. S. Fradkin and M. A. Vasiliev, Nucl. Phys. B291 (1987) 141, Annals Phys. 177 (1987) 63, Phys. Lett. B189 (1987) 89, JETP Lett. 44 (1986) 622 [Pisma Zh. Eksp. Teor. Fiz. 44 (1986) 484], Int. J. Mod. Phys. A3 (1988) 2983.

[23] A. Sagnotti, E. Sezgin and P. Sundell, arXiv:hep-th/0501156.

[24] M. Dubois-Violette and M. Henneaux, [arXiv:math.qa/9907135], Commun. Math. Phys. **226** (2002) 393 [arXiv:math.qa/0110088].

[25] M. Bianchi and V. Didenko, arXiv:hep-th/0502220.

[26] D. Francia and C. M. Hull, arXiv:hep-th/0501236.

[27] N. Bouatta, G. Compere and A. Sagnotti, arXiv:hep-th/0409068.

[28] X. Bekaert, S. Cnockaert, C. Iazeolla and M. A. Vasiliev, arXiv:hep-th/0503128.

[29] D. Francia, J. Mourad and A. Sagnotti, to appear.

[30] X. Bekaert and N. Boulanger, Phys. Lett. B **561** (2003) 183 [arXiv:hep-th/0301243].

[31] P. de Medeiros and C. Hull, JHEP **0305** (2003) 019 [arXiv:hep-th/0303036].

[32] M. A. Vasiliev, Fortsch. Phys. **35** (1987) 741 [Yad. Fiz. **45** (1987) 1784]; V. E. Lopatin and M. A. Vasiliev, Mod. Phys. Lett. A **3** (1988) 257; M. A. Vasiliev, Nucl. Phys. B **301** (1988) 26.

[33] E. Sezgin and P. Sundell, JHEP **0207** (2002) 055 [arXiv:hep-th/0205132].